Power law diffusion coefficient and anomalous diffusion: Analysis of solutions and first passage time

Kwok Sau Fa and E. K. Lenzi
Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá-PR, Brazil

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We investigate one-dimensional equations for the diffusion with a nonconstant diffusion coefficient inside the second derivative and between the derivatives. In particular, we employ the diffusion coefficient $D(x) \propto |x|^{-\theta}$ ($\theta \in \mathbf{R}$) and a quartic potential. These diffusion equations present a rich variety of behaviors associated with different regimes. Results of two approaches are analyzed and compared. We also investigate the mean first passage time of these systems. We show that the system with the coefficient $D(x)$ between the derivatives can produce different behaviors for the mean first passage time in comparison with those obtained by the system with the coefficient inside the derivatives.

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I. INTRODUCTION

The Kramers problem or the problem of surmounting a potential barrier is undoubtedly one of the most important themes in physics in connection with several topics [1]. For example, it plays a key role in stochastic resonance [2], in describing fluctuation-induced transport such as it occurs in kink motion [3] and ratchets [4]. Even the extent of chaos in the Hamiltonian systems has been shown to have connections with this quantity [5]. A collection of these and other stochastically driven processes can be found in Refs. [6,7]. In this scenario, we also have a quantity known as the mean first passage time (or the escape time), directly related to the Kramers problem, which is defined as the time T when the process starts from a given point and reaches a predetermined level for the first time.

It should be noted that the mean first passage time has been analyzed in many systems where the Brownian motion ($\langle x^2\rangle \propto t$) is present. Recently, the study of the mean first passage time in other contexts, such as the systems that exhibit anomalous diffusion, has attracted a lot of attention [8]. Further, we can mention the works [9] where the mean first passage time has been analyzed in the fractional derivative approach, and in Ref. [10] the escape time has been discussed in terms of a nonlinear Fokker-Planck equation. For these cases, the diffusion is anomalous correlated type, i.e., the second moment is defined and is given by $\langle x^2\rangle \propto t^{\alpha}$ in contrast with the usual case $\alpha=1$. For $0<\alpha<1$ and $\alpha>1$, the system describes subdiffusive and superdiffusive processes, respectively. Similar anomalous behavior may also be obtained in the linear diffusion equation by taking a spatial dependence in the diffusion coefficient into account, for example, $D(x)\propto|x|^{-\theta}$. In particular, this coefficient has been applied to several physical situations such as fast electrons in a hot plasma in the presence of dc electric field [11], turbulent two-particle diffusion in configuration space [12]; also, both the Richardson [13] and the Kolmogorov [14] laws applied it to turbulence and to describe the diffusion on fractals [15].

In this work we analyze two forms of equations (for diffusion) which can describe anomalous diffusion. They are distinguished by a nonconstant diffusion coefficient $D(x) \propto |x|^{-\theta}$ inside the second derivative and between the derivatives. We also analyze the mean first passage time of these systems with the presence of a quartic potential. This way, we can study the properties concerning the escape time for the systems whose diffusive anomalous processes are present. Also, we can compare the behaviors of systems of different approaches. In particular, an understanding of the mean first passage time in such systems could open up comprehension of new stochastically driven phenomena.

This work is divided into four sections. In Sec. II, we consider the diffusion equations without the presence of an external force and we present their solutions for the probability distribution and the second moment. Also, different regimes are presented and discussed. In Sec. III, we consider the diffusion equations with the presence of an external potential, and, in particular, a quartic potential. We obtain the stationary solutions, and their behaviors are analyzed. We study the mean first passage time of these systems and compare the results of two approaches as well. In Sec. IV, we present our conclusions.

II. DIFFUSION EQUATIONS

We first consider the diffusion systems without the presence of any external force. For simplicity, we only study one-dimension systems. As motivated in the preceding section we shall analyze the following diffusion equations:

$$\frac{\partial \rho_1}{\partial t} = D\frac{\partial^2 |x|^{-\theta}\rho_1}{\partial x^2} \tag{1}$$

and

$$\frac{\partial \rho_2}{\partial t} = D\frac{\partial}{\partial x}\left[|x|^{-\theta}\frac{\partial \rho_2}{\partial x}\right], \tag{2}$$

where $D|x|^{-\theta}$ is the diffusion coefficient. Equation (1) is a Fokker-Planck equation, whereas Eq. (2) is a one-dimensional diffusion equation which has been studied, for instance, by O'Shaughnessy and Procaccia based on fractal geometry [15]. Equation (2) also corresponds to a Fokker-

Planck equation in a different order of prescription in stochastic calculus (see Bouchaud and Georges [8] and [16,17]). Usually, the form of Eq. (2) is known as the transport form of the Fokker-Planck equation [17]. The transport form in Eq. (2) results from using a stochastic calculus description which is different from the Ito and the Stratonovich rule; Eq. (2) is the result of the postpoint discretization rule for the stochastic integral prescription [18].

As we shall see, the above equations can describe the localized, superdiffusive, normal diffusive, and subdiffusive processes. We note that Eqs. (1) and (2) are symmetric under the change $x\rightarrow -x$. The physical solution of Eq. (1) can be obtained as follows. By using the following transformation,

$$z=\frac{x}{\Phi(t)} \tag{3}$$

and setting

$$\rho_1=\frac{f\left(\frac{x}{\Phi(t)}\right)}{\Phi(t)}, \tag{4}$$

Eq. (1) is transformed to

$$-\frac{\dot{\Phi}(t)}{\Phi^2(t)}\frac{d}{dz}[zf(z)]=D[\Phi(t)]^{-3-\theta}\frac{d^2}{dz^2}[|z|^{-\theta}f(z)]. \tag{5}$$

Now, we can obtain a set of solutions by separating the variables t and z, and putting

$$\dot{\Phi}(t)[\Phi(t)]^{1+\theta}=k, \tag{6}$$

where k is a constant. The solution for $\Phi(t)$ is given by

$$\Phi(t)=[k(2+\theta)t]^{1/(2+\theta)}. \tag{7}$$

For convenience, we have taken the integration constant of solution (7) equal to zero. The other equation that depends on the variable z is given by

$$-k\frac{d}{dz}[zf(z)]=D\frac{d^2}{dz^2}[|z|^{-\theta}f(z)]. \tag{8}$$

This last equation has the following solution:

$$f(z)=f_0|z|^{\theta}\exp\left[-\frac{k|z|^{2+\theta}}{D(2+\theta)}\right], \tag{9}$$

where f_0 is one of the integration constants and the other integration constant was put equal to zero.

The normalized solution for ρ_1 is given by

$$\rho_1=C\frac{|x|^{\theta}\exp\left[-\frac{|x|^{2+\theta}}{D(2+\theta)^2t}\right]}{[Dt]^{(1+\theta)/(2+\theta)}}, \tag{10}$$

where C is a constant to be determined by the normalization of ρ_1. For $\theta<-2$ and $\theta>-1$, C is given by $C=1/2\Gamma(1+\theta/2+\theta)|2+\theta|^{\theta/(2+\theta)}$. For $-2<\theta\leqslant-1$, the distribution function ρ_1 is not normalizable. We should note that the distribution ρ_1 diverges at $x=0$ in the interval $-2\leqslant\theta<0$, from which the system is locked at the singular point. We see that ρ_1 has the form of a stretched Gaussian multiplied by $|x|^{\theta}$. For $\theta=0$, we recover the Wiener process and the distribution recovers the usual Gaussian form. The second moment, for $\theta<-3$ and $\theta>-1$, yields

$$\langle x^2\rangle=\frac{\Gamma\left[\frac{3+\theta}{2+\theta}\right]}{\Gamma\left[\frac{1+\theta}{2+\theta}\right]}[D(2+\theta)^2t]^{2/(2+\theta)}. \tag{11}$$

The result (11) shows that system (1) can describe the localized processes ($\theta<-3$), superdiffusive processes ($-1<\theta<0$), normal diffusive processes ($\theta=0$), and subdiffusive processes ($\theta>0$). For $-3\leqslant\theta<-2$, the second moment is divergent. In this last case, the regime presents a similar characteristic to the Lévy process.

In the case of system (2) the solution was obtained in Refs. [15,19] and is given by

$$\rho_2=\frac{1}{2\Gamma\left(\frac{3+\theta}{2+\theta}\right)}\frac{\exp\left[-\frac{|x|^{2+\theta}}{D(2+\theta)^2t}\right]}{[D(2+\theta)^2t]^{1/(2+\theta)}}. \tag{12}$$

The factor of 2 in the denominator is absent in the expression given in Ref. [15] due to the fact that we have considered the whole space. This distribution also has the form of a stretched Gaussian. For $\theta=0$, we recover the Wiener process and the distribution ρ_2 recovers the usual Gaussian form. The second moment of this system is

$$\langle x^2\rangle=\frac{\Gamma\left[\frac{3}{2+\theta}\right]}{\Gamma\left[\frac{1}{2+\theta}\right]}[D(2+\theta)^2t]^{2/(2+\theta)}. \tag{13}$$

This result shows that system (2) can describe the following processes: superdiffusive $-2<\theta<0$, normal diffusive $\theta=0$, and subdiffusive $\theta>0$. For $\theta<-2$, the distribution function ρ_2 is not normalizable. The validity of solution (12) is restricted to the interval $\theta>-2$.

The distribution (10), in particular $\rho_1|x|^{-\theta}$, has a similar form to that obtained in Eq. (12) using the diffusion equation (2). However, the term $|x|^{\theta}$, which appears in Eq. (10), splits the peak of the stretched Gaussian into two ones, and the distribution ρ_1 presents similar characteristic to the asymptotic distribution on fractals, such as the Sierpinski gasket [20]. As in Ref. [15], θ can be identified as the exponent of the anomalous diffusion in both the systems. We see that two different systems can give the same form for the second moment, except for the coefficients. It should be

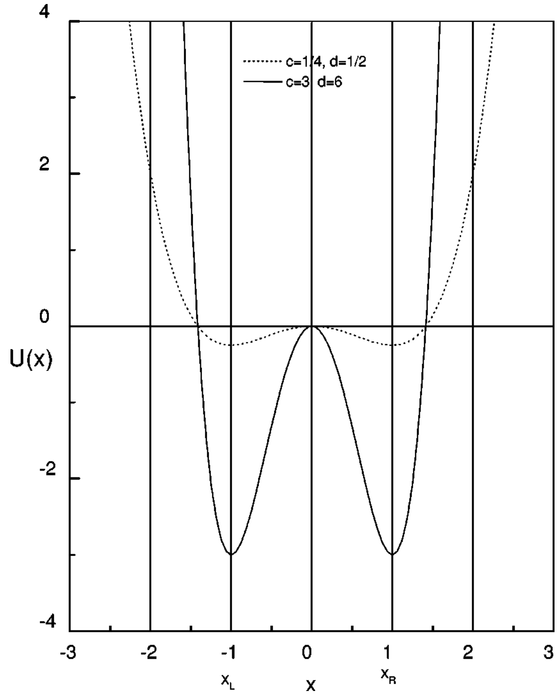

FIG. 1. Plots of the quartic potential $U(x)=cx^4-dx^2$ in arbitrary units.

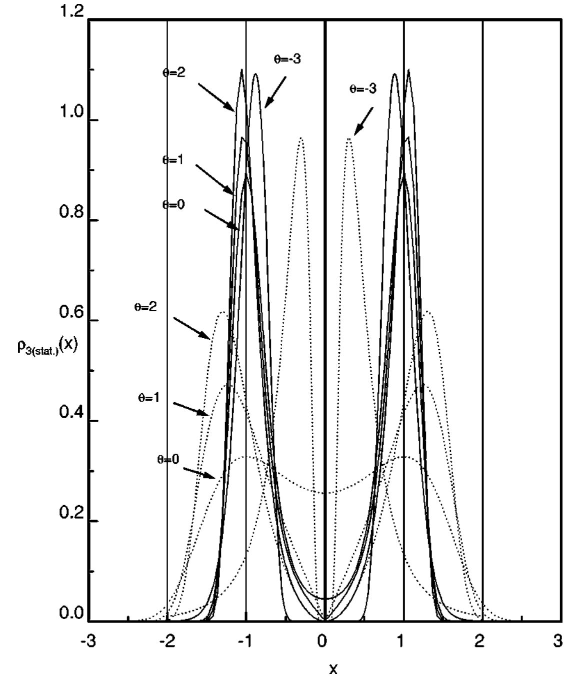

FIG. 2. Plots of the normalized stationary distribution (16) described by Eq. (14), in arbitrary units. The solid lines correspond to the potential with $c=3$, $d=6$, and $D=1$. The dotted lines correspond to the potential with $c=1/4$, $d=1/2$, and $D=1$.

noted that Eq. (2) can be transformed in the form of equation (1). This way, Eq. (2) can be viewed as the Fokker-Planck equation (1) with an external potential, and the divergence, which appears in the distribution ρ_1 of the system for $-2<\theta<0$, described by the Fokker-Planck equation (1), is removed by this external potential. Further discussion on this subject will be given in the following section.

III. MEAN FIRST PASSAGE TIME

Here, we consider the diffusion equations presented in the preceding section with the presence of an external force, which are given by

$$\frac{\partial\rho}{\partial t}=\frac{\partial[U'(x)\rho]}{\partial x}+D\frac{\partial^2|x|^{-\theta}\rho}{\partial x^2} \tag{14}$$

and

$$\frac{\partial\rho}{\partial t}=\frac{\partial[U'(x)\rho]}{\partial x}+D\frac{\partial}{\partial x}\left[|x|^{-\theta}\frac{\partial\rho}{\partial x}\right], \tag{15}$$

where $U(x)$ is a time-independent confining potential. In particular, we consider the familiar quartic potential $U(x)=cx^4-dx^2$, where c and d are positive real constants (see, for example, Ref. [21]). In Fig. 1 we show the form of the quartic potential for ($c=1/4$, $d=1/2$) and ($c=3$, $d=6$), where it can be noted that the second pair gives a greater potential height. In these systems we are interested in the investigation of the diffusion of a particle over a barrier, namely, the mean first passage time. Moreover, these systems permit us to analyze, at the same time, the behaviors of the mean first passage time in different regimes and to compare them. Analytical solutions of these systems cannot be easily obtained. Thus, we only obtain the stationary solutions. The stationary solutions can give us some insights into the systems and, therefore, the behaviors of the first passage time can be better understood. The stationary solutions, with natural boundary conditions [22], of Eqs. (14) and (15) are given, respectively, by

$$\rho_{3(stat.)}=c_1|x|^{\theta}\exp\left[-\frac{4c|x|^{4+\theta}}{D(4+\theta)}+\frac{2d|x|^{2+\theta}}{D(2+\theta)}\right] \tag{16}$$

and

$$\rho_{4(stat.)}=c_2\exp\left[-\frac{4c|x|^{4+\theta}}{D(4+\theta)}+\frac{2d|x|^{2+\theta}}{D(2+\theta)}\right], \tag{17}$$

where c_1 and c_2 are constants to be determined by the normalization.

In Figs. 2 and 3, we show the plots of the normalized distributions given by Eqs. (16) and (17). It is interesting to note that the peaks of the stationary distributions (16) are dislocated to the walls of the potential with the increase of the values of θ (Fig. 2). This means that the particles are pushed against the walls of the potential for $\theta>0$. For $-3<\theta<-2$, the distribution keeps close to the origin $x=0$ and the barrier of the potential tends to separate the peaks of distribution. Also, the height of the peaks increases with the decrease of the value of θ, whereas the behaviors of stationary solution (17) described by diffusion equation (15) are different. The peaks of distributions (Fig. 3) are kept at the minima of the potential with different values of θ and the parameters of the potential (c and d). These results can be

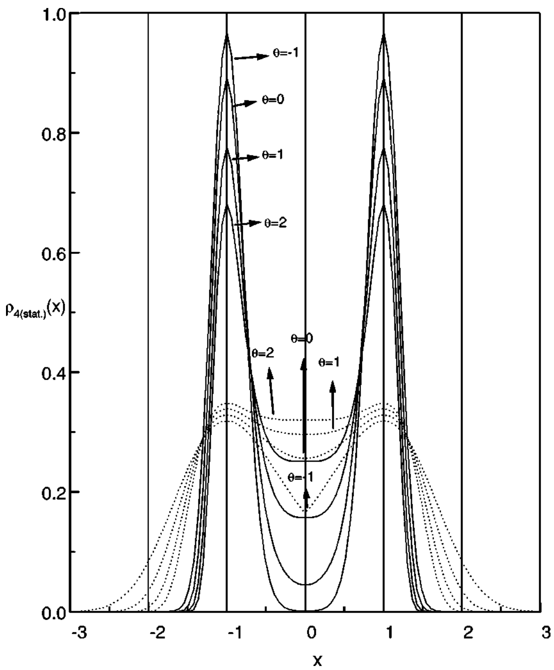

FIG. 3. Plots of the normalized stationary distribution (17) described by Eq. (15), in arbitrary units. The solid lines correspond to the potential with $c=3$, $d=6$, and $D=1$. The dotted lines correspond to the potential with $c=1/4$, $d=1/2$, and $D=1$.

understood as follows. Equation (15) can be written as a Fokker-Planck equation of the type (14)

$$\frac{\partial \rho}{\partial t}=\frac{\partial}{\partial x}\left[\frac{d[U(x)-D|x|^{-\theta}]}{dx}\rho\right]+D\frac{\partial^2|x|^{-\theta}\rho}{\partial x^2}. \tag{18}$$

This equation has the potential

$$\overline{U}(x)=U(x)-D|x|^{-\theta}. \tag{19}$$

The term $D|x|^{-\theta}$ becomes the potential $\overline{U}(x)$ deeper than $U(x)$ and it modifies the potential $U(x)$ for $\theta>0$ greatly. Due to this fact the particles are attracted more to the center, and the peaks of the distributions are maintained at the minima of potential.

Now we analyze the mean first passage time of Eq. (14). To do so, we present the formulas that we shall use for our calculation. A closed expression for the mean first passage time which relates the coefficients of the Fokker-Planck equation can be obtained by the Backward equation [1,16], i.e.,

$$T(x)=\int_x^b \frac{dy}{\psi(y)}\int_a^y \frac{\psi(z)}{B(z)}dz \tag{20}$$

and

$$\psi(x)=\exp\left\{\int_a^x \frac{A(z)}{B(z)}dz\right\}, \tag{21}$$

with the boundary condition: with a reflecting, b absorbing, and $a<b$. The coefficients, for system (14), are related to $A(z)=-U'(z)$ and $B(z)=D\ |z|^{-\theta}$. In this case, Eq. (20) should be evaluated numerically. Our calculation is performed on an infinite range with the substitutions in the limits of integrals by $b\rightarrow x_0$, $a\rightarrow-\infty$, and $x\rightarrow x_L$, where x_L is the left minimum of the potential and x_0 is the point in which the particle is removed from the system [16]. Therefore, $T(x_L\rightarrow x_0)$ represents a measure of the escape time from the left-hand to the right-hand well.

In order to confirm our result, we also perform numerical calculation by using, in parallel, the corresponding Langevin equation given by

$$x_{n+1}=x_n+A(x_n,t_n)\tau+\sqrt{B(x_n,t_n)\tau}w_n, \tag{22}$$

where the time interval (Δt) is divided into N small finite steps of length $\tau=\Delta t/N$ and w_n is a random number generator [22]. We should note that Eq. (22) cannot be used for $\theta>0$ due to the singularity of $B(z)$ at the origin $(x=0)$, but we can use it to check out our result for $\theta\leqslant 0$. In the calculation of Eq. (22) we have performed 2000 realizations and then taken the average time to obtain the mean first passage time. The agreement between two approaches [Eqs. (20) and (22)] is good.

In Fig. 4(a), we show the mean first passage time with respect to x for different regimes with the parameters of the potential given by $c=1/4$, $d=1/2$, and $D=1$. In these cases there is no formation of plateau due to the fact that the height of potential barrier is small. Further, we can note that there are "competitions" concerning $T(x_L\rightarrow x_0)$ for different regimes. Initially, the smaller the value of θ, the faster the escape time $T(x_L\rightarrow x_0)$. Then, the processes are inverted, and finally, close to the right potential wall, the processes are inverted again. When the height of potential barrier increases, the plateau becomes evident as it can be seen from Fig. 4(b). For $\theta=-1/2$, we see that the mean first passage time takes a long time before crossing the barrier, but it does not diverge. In this last case, the particles tend to accumulate close to the origin (see Fig. 2). For $\theta<-2$, $T(x_L\rightarrow x_0)$ diverges before crossing the barrier. Moreover, we note that the subdiffusive regimes run faster than those of normal regimes to surmount the barrier. After surmounting the barrier, the quantity $T(x_L\rightarrow x_0)$ of normal regime, at a certain point of x, becomes faster than those of subdiffusive regimes. In Fig. 4(c), we see again the "competitions" concerning $T(x_L\rightarrow x_0)$ for different regimes. The value of the noise strength D at the intersection points, for different regimes, depends on the height of the potential barrier.

The mean first passage time of system (15) can also be obtained by using formulas (20) and (21). In this case, we should consider Eq. (15) in the form of the Fokker-Planck equation (14), i.e., the Backward equation of Eq. (18) with the potential $\overline{U}$ given by Eq. (19) and the diffusion coefficient $D|x|^{-\theta}$. The coefficients $A(z)$ and $B(z)$ are given by $A(z)=-d[\overline{U}(z)]/dz$ and $B(z)=D|z|^{-\theta}$. From Figs. 5(a) and 5(b) we note that the behaviors of the mean first passage time (for $\theta>0$) are very different from those obtained by the Fokker-Planck equation (14) [see Figs. 4(a) and 4(b)]. They are modified, mainly, in the region where the term $D|x|^{-\theta}$

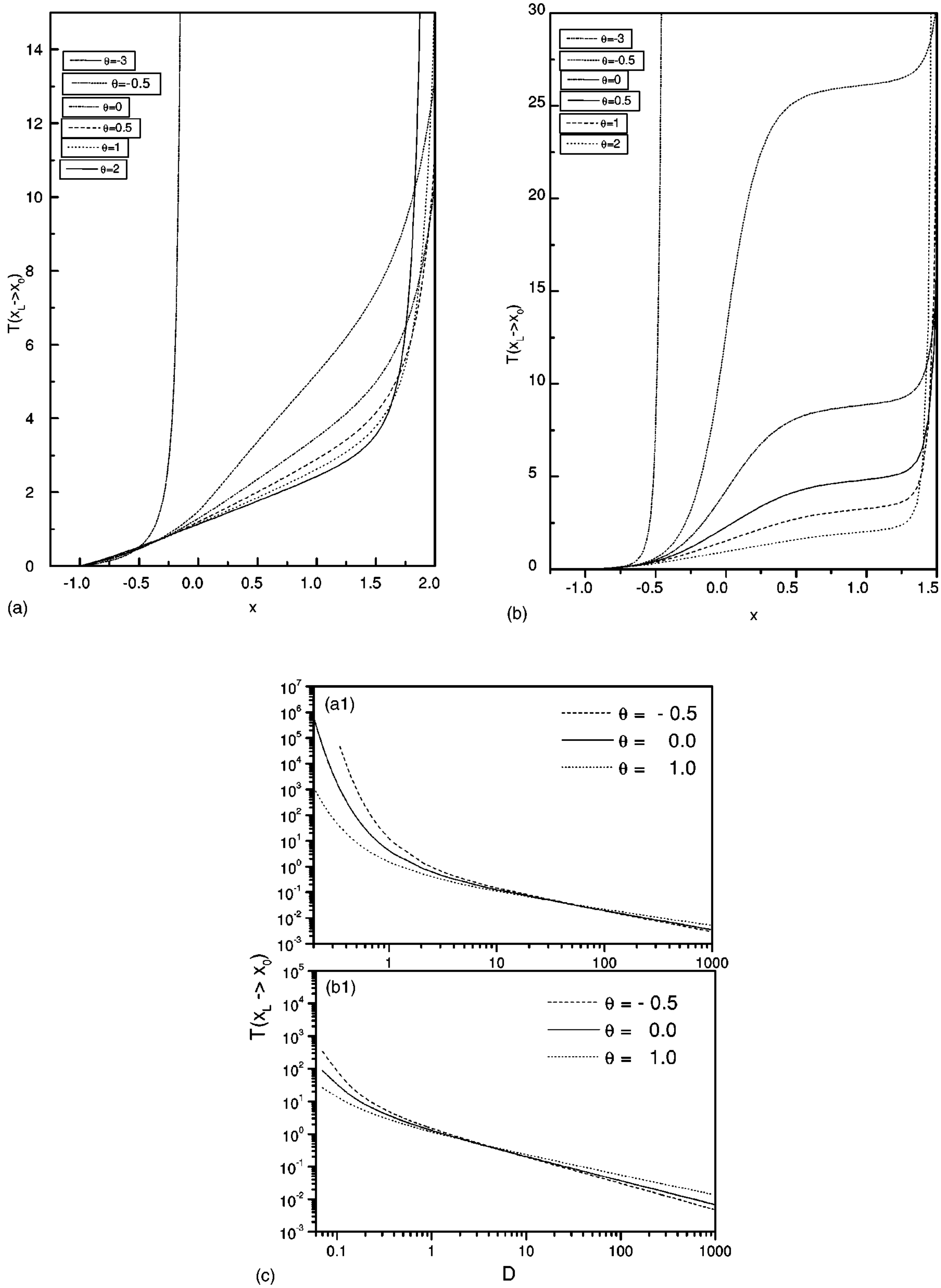

FIG. 4. (a) Plots of the mean first passage time of system (14) for different regimes with $c=1/4$, $d=1/2$, and $D=1$, in arbitrary units. (b) Plots of the mean first passage time of system (14) for different regimes with $c=3$, $d=6$, and $D=1$, in arbitrary units. (c) Plots of the mean first passage time of system (14) in function of D for different regimes, in arbitrary units. (a1) For $c=3$ and $d=6$. (b1) For $c=1/4$ and $d=1/2$. $T(x_L \rightarrow x_0)$ is calculated from x_L to the local maximum of $U(x)$ at $x=0$.

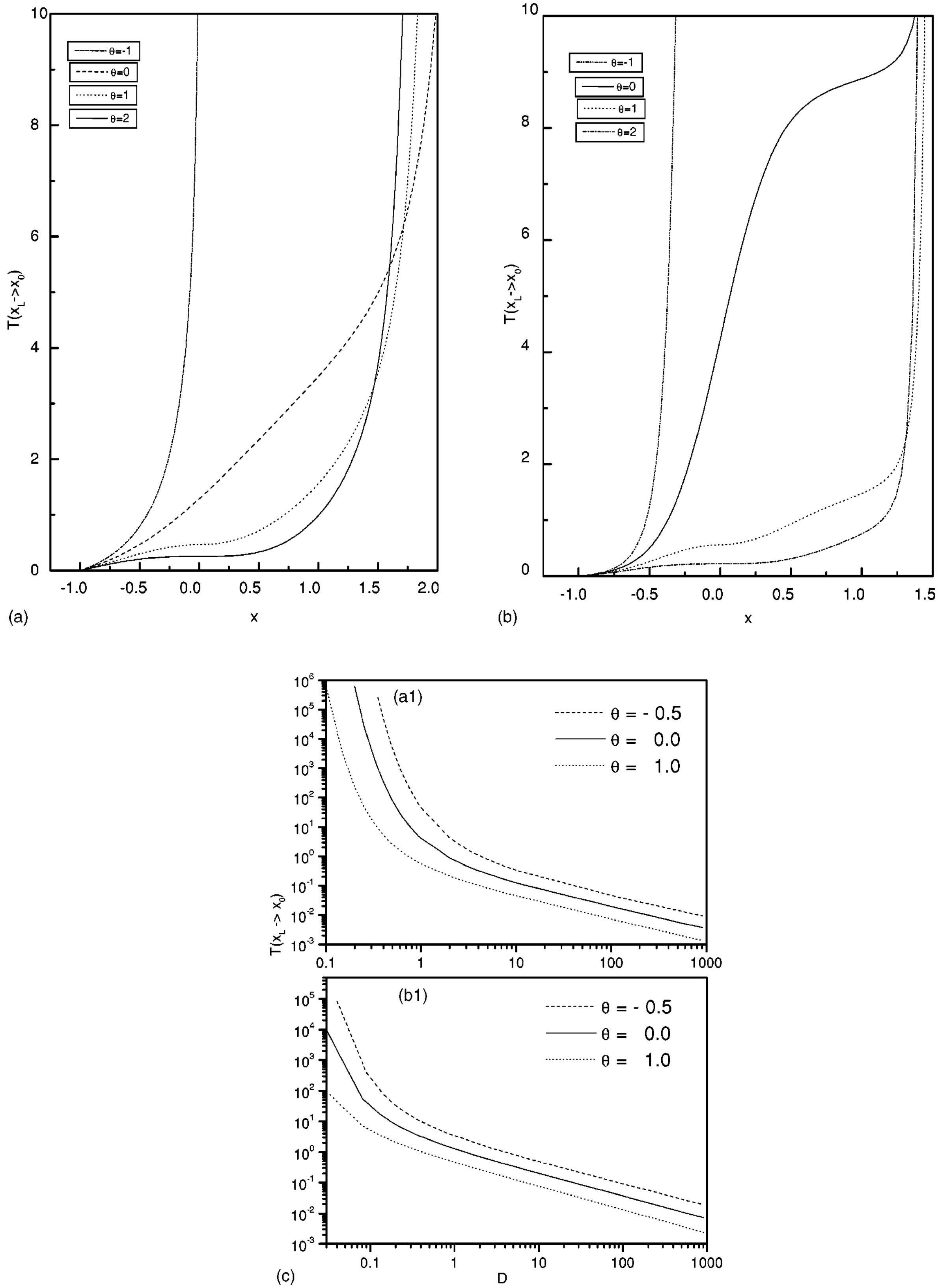

FIG. 5. (a) Plots of the mean first passage time of system (15) for different regimes with $c=1/4$, $d=1/2$, and $D=1$, in arbitrary units. (b) Plots of the mean first passage time of system (15) for different regimes with $c=3$, $d=6$, and $D=1$, in arbitrary units. (c) Plots of the mean first passage time of system (15) in function of D for different regimes, in arbitrary units. (a1) For $c=3$ and $d=6$. (b1) For $c=1/4$ and $d=1/2$. $T(x_L \rightarrow x_0)$ is calculated from x_L to the local maximum of $U(x)$ at $x=0$.

has the significative contribution to the new potential $\overline{U}$. As we can verify, the particle spends almost no time, for $\theta > 0$, to cross the interval near the origin ($x=0$). Contrary to the Fokker-Planck equation (14), for $\theta>0$, the plateau is formed even though the height of potential barrier of $U(x)$ is small. This behavior can be understood from the transformed potential $\overline{U}$ which has an infinite well centered at the origin $x=0$. In general, for $\theta>0$, $T(x_L \rightarrow x_0)$ of the diffusion equation (15), plotted in Figs. 5(a) and 5(b), is smaller than that of Eq. (14) [Figs. 4(a) and 4(b)]. For $\theta=-1$, the mean first passage time corresponding to Eq. (15) diverges before crossing the barrier of $U(x)$. Figure 5(c) shows $T(x_L \rightarrow x_0)$ in function of D. In this case, there are no "competitions" concerning $T(x_L \rightarrow x_0)$ for different regimes for a large range of D. The behaviors of the escape time for anomalous regimes are similar to that of the normal regime ($\theta=0$).

IV. CONCLUSION

We have studied two forms of diffusion equations: Eqs. (1) and (2). As a matter of fact, there are many different descriptions; for instance, some of them, including the two forms studied in this work, have been analyzed and applied to diverse systems [16,18,23]. In particular, Eq. (2) may be useful to investigate fractal behavior in gravitational systems [24]. We have shown that the diffusion coefficient $D|x|^{-\theta}$ can also produce a stretched Gaussian distribution in the framework of the Fokker-Planck equation. Although the diffusion equation (2) has been conceived by O'Shaughnessy and Procaccia [15], with an appeal for the diffusion on fractal structures, the solution of Fokker-Planck equation (10) approximates more to the asymptotic distribution on fractals, such as the Sierpinski gasket (see Metzler and Klafter [9] and [20]) than that obtained by the diffusion equation (12). We have shown that these diffusion equations can describe several kinds of anomalous diffusion processes: subdiffusive, superdiffusive, localized, and the Lévy type. Our comparative study on the behaviors of mean first passage time of these systems have shown that they are very different. As we have noted, the diffusion equation (15) can be written in the form of the Fokker-Planck equation (14) plus an attractive potential. This potential term has the tendency of attracting the particles into the regions close to the origin.

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